SIMPLIFICATION OF ENGINEERING CALCULATIONS FOR ROTATIONAL BODIES USING PAPPUS-GULDINUS THEOREMS

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Abstract: *In this article, the applications of the Pappus-Guldinus theorems in simplifying the calculations of surface areas and volumes of rotational bodies are investigated. Traditional methods using multiple integrals are compared with the elegant geometric approach provided by these theorems. To demonstrate the efficiency and accuracy of these theorems in modern engineering and applied mechanics, several practical problems - including the parameters of a torus, a right cone, and a sphere - are presented with step-by-step solutions.*

Keywords: *Pappus-Guldinus theorems, rotational bodies, center of mass, multiple integrals, engineering calculations, torus volume, surface area.*

Calculating the surface area and volume of complex three-dimensional objects is a fundamental task in engineering, architecture, and applied mechanics. While the advent of integral calculus provided a rigorous mathematical framework for these calculations, evaluating multiple integrals can often be computationally intensive and prone to algebraic errors. In many

practical scenarios, especially when dealing with rotational bodies, the Pappus-Guldinus theorems offer a powerful and simplified alternative. These classical theorems elegantly bridge the gap between pure geometry and integral calculus by utilizing the properties of the center of mass.

The primary purpose of this article is to demonstrate the efficiency of the Pappus-Guldinus theorems in solving complex engineering problems, comparing them with standard integral methods, and providing clear, step-by-step examples across different geometric shapes.

The theorems, attributed to Pappus of Alexandria and later refined by Paul Guldin, are formulated as follows:

First Theorem (Surface Area): The surface area S of a surface of revolution generated by rotating a plane curve L about a non-intersecting axis in its plane is equal to the product of the arc length of the curve l and the distance traveled by its geometric centroid (center of mass) during one full rotation.

Mathematically, it is expressed as:

$$S = l \cdot 2\pi R_c$$

where l is the arc length of the curve, and R_c is the perpendicular distance from the centroid of the curve to the axis of rotation.

Second Theorem (Volume): The volume V of a solid of revolution generated by rotating a plane figure D about a non-intersecting axis in its plane is equal to the product of the area of the figure A and the distance traveled by its geometric centroid during one full rotation.

Mathematically, it is given by:

$$V = A \cdot 2\pi R_c$$

where A is the area of the plane figure.

To illustrate the universality of the Pappus-Guldinus theorems, let us examine practical calculations for three different types of rotational bodies.

Example 1: Volume and Surface Area of a Torus

The torus shape is widely used in the design of hydraulic seals (O-rings) and magnetic confinement structures (tokamaks) (Figure 1.).

Problem: Calculate the surface area and volume of a torus formed by rotating a circle of radius r around a coplanar axis located at a distance R from its center, where $R > r$.

Solution:

1. The generating figure is a circle.

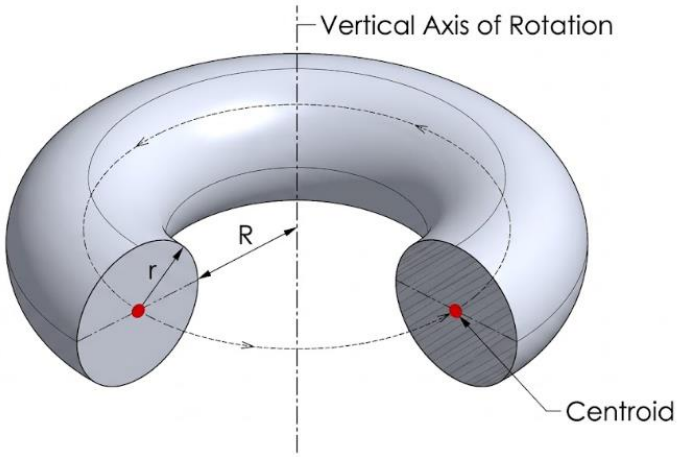


Figure 1.

Area of the circle: $A = \pi r^2$

Circumference of the curve: $l = 2\pi r$

2. The centroid of a circle is located exactly at its geometric center. Therefore, the distance from the centroid to the axis of rotation is $R_c = R$.

3. According to the First Theorem, the surface area S is:

$$S = l \cdot 2\pi R_c = (2\pi r) \cdot (2\pi R) = 4\pi^2 rR$$

1. According to the Second Theorem, the volume V is:

$$V = A \cdot 2\pi R_c = (\pi r^2) \cdot (2\pi R) = 2\pi^2 r^2 R$$

By avoiding complex multiple integrals, this method provides an exact analytical solution instantly.

Example 2: Volume and Lateral Surface Area of a Right Cone

Cone-shaped components, often found in architecture and mechanical engineering, are formed by rotating a right-angled triangle.

Problem: Find the volume and lateral surface area of a right cone generated by rotating a right-angled triangle with base r and height h around its height (vertical leg).

Solution (Volume):

1. The generating figure is a right-angled triangle. Its area is $A = \frac{1}{2}rh$.

2. From geometry, the centroid of a triangle lies at the intersection of its medians. Its distance from the vertical axis of rotation is $R_c = \frac{r}{3}$.

3. Applying the Second Theorem:

$$V = A \cdot 2\pi R_c = \left(\frac{1}{2}rh\right) \cdot 2\pi \left(\frac{r}{3}\right) = \frac{1}{3}\pi r^2 h$$

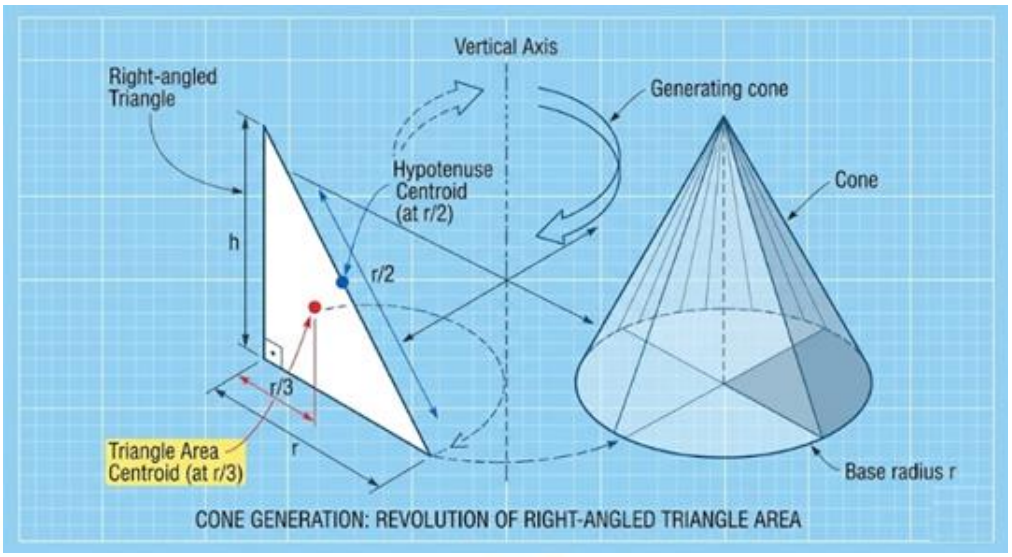


Figure 2.

Solution (Lateral Surface Area):

1. The lateral surface is generated by rotating the hypotenuse. The length of the hypotenuse is $l = \sqrt{r^2 + h^2}$.

2. The centroid of a straight line segment is at its midpoint. Thus, the distance from the midpoint of the hypotenuse to the vertical axis is half the base, $R_c = \frac{r}{2}$.

3. Applying the First Theorem:

$$S_{lateral} = l \cdot 2\pi R_c = \sqrt{r^2 + h^2} \cdot 2\pi \left(\frac{r}{2} \right) = \pi r \sqrt{r^2 + h^2}$$

If we denote the slant height $\sqrt{r^2 + h^2}$ as L, we obtain the familiar formula $S_{lateral} = \pi r L$.

Example 3: Volume and Surface Area of a Sphere

Determining the parameters of a sphere is one of the most common tasks in hydrodynamics for studying vessel capacities.

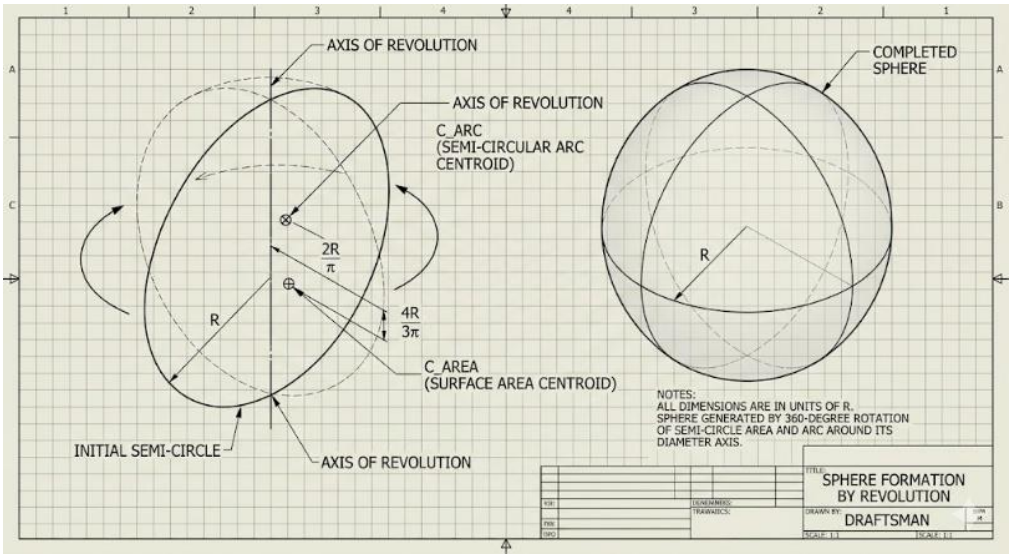


Figure 3.

Problem: Calculate the volume and surface area of a sphere generated by rotating a semi-circle of radius R around its diameter.

Solution:

1. The area of the semi-circle is $A=\frac{\pi R^2}{2}$. From integral calculus, the centroid of a semi-circle is located at a distance $R_c=\frac{4R}{3\pi}$ from its diameter.

2. Based on the Second Theorem, the volume of the sphere is:

$V=A\cdot 2\pi R_c=\left(\frac{\pi R^2}{2}\right)\cdot 2\pi\left(\frac{4R}{3\pi}\right)=\frac{4}{3}\pi R^3$ To find the surface area, we rotate the semi-circular arc. The length of the arc is $l=\pi R$. Its centroid lies at a distance $R_c=\frac{2R}{\pi}$ from the diameter.

3. According to the First Theorem, the surface area of the sphere is:

$$S=l\cdot 2\pi R_c=(\pi R)\cdot 2\pi\left(\frac{2R}{\pi}\right)=4\pi R^2$$

The Pappus-Guldinus theorems remain an indispensable tool in the arsenal of modern engineers and applied mathematicians. By leveraging the geometric properties of centroids, these theorems reduce the complexity of multi-dimensional integral calculations into straightforward arithmetic products. The integration of such classical geometric principles into contemporary engineering education and computational design systems enhances both accuracy and efficiency.

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