EVALUATING THE WORK OF VARIABLE FORCES AND HYDROSTATIC PRESSURE USING DEFINITE INTEGRALS

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Abstract: *This article explores the vital application of definite integrals in solving complex physical and engineering problems, specifically focusing on evaluating the work done by variable forces and calculating hydrostatic pressure on submerged surfaces. Traditional algebraic methods fail when dealing with continuously changing parameters. By applying integral calculus, precise analytical solutions can be derived for non-linear physical phenomena. Practical examples, including Hooke's law for springs, isothermal gas expansion, and fluid pressure on a dam, are presented to demonstrate the mathematical modeling of these physical processes.*

Keywords: *Definite integral, variable force, hydrostatic pressure, Hooke's law, fluid mechanics, thermodynamics, applied mathematics, engineering calculations.*

In applied mechanics and physics, many fundamental quantities such as work, fluid pressure, and center of mass cannot be calculated using simple multiplication formulas because the underlying factors (force, pressure) are not constant. When a force varies continuously over a distance, or when fluid pressure increases continuously with depth, the definite integral becomes the

essential mathematical tool to accumulate these infinitely small changes into an exact macroscopic value.

The primary purpose of this article is to bridge theoretical calculus and applied physics by demonstrating how definite integrals are utilized to evaluate the work of variable forces and hydrostatic pressure, providing clear, step-by-step engineering examples.

In classic mechanics, if a constant force F moves an object along a straight line by a distance d, the work done is simply $W = F \cdot d$. However, if the force varies as a function of position x, denoted as $F(x)$, the total work done in moving the object from $x = a$ to $x = b$ must be defined as a definite integral:

$$W = \int_a^b F(x)dx$$

Example 1: Stretching a Spring (Hooke's Law)

According to Hooke's Law, the force required to maintain a spring stretched by x units beyond its natural length is proportional to x. Thus, $F(x) = kx$, where k is the spring constant (Figure 1.).

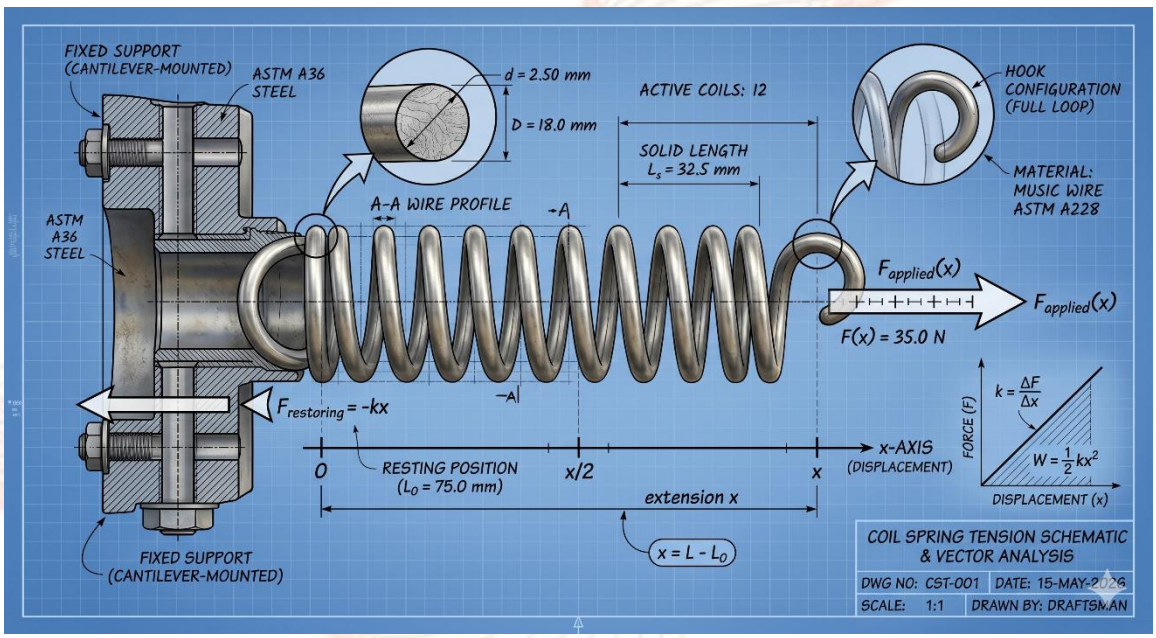

Figure 1.

Problem: A force of 40 N is required to hold a spring stretched 0.2 meters beyond its natural length. How much work is done in stretching the spring from its natural length to 0.5 meters?

Solution:

1. First, we find the spring constant k using the given condition:

$F(0.2) = 40 \Rightarrow k(0.2) = 40 \Rightarrow k = 200\text{N/m}$

2. The force function is $F(x) = 200x$.

3. The work done to stretch the spring from $x = 0$ to $x = 0.5$ is evaluated using the definite integral:

$$W = \int_0^{0.5} 200x\,dx = 200\left[\frac{x^2}{2}\right]_0^{0.5} = 100(0.5^2) = 100(0.25) = 25\text{Joules}$$

This demonstrates how integration seamlessly handles linearly increasing forces.

Example 2: Work in Thermodynamics (Expanding Gas)

When a gas expands in a cylinder, it exerts a variable pressure on a piston. The work done by the gas when its volume expands from V_1 to V_2 is given by:

$W = \int_{V_1}^{V_2} P\,dV$ For an isothermal process (constant temperature), Boyle's Law states that $P \cdot V = C$ (where C is a constant), so $P(V) = \frac{C}{V}$.

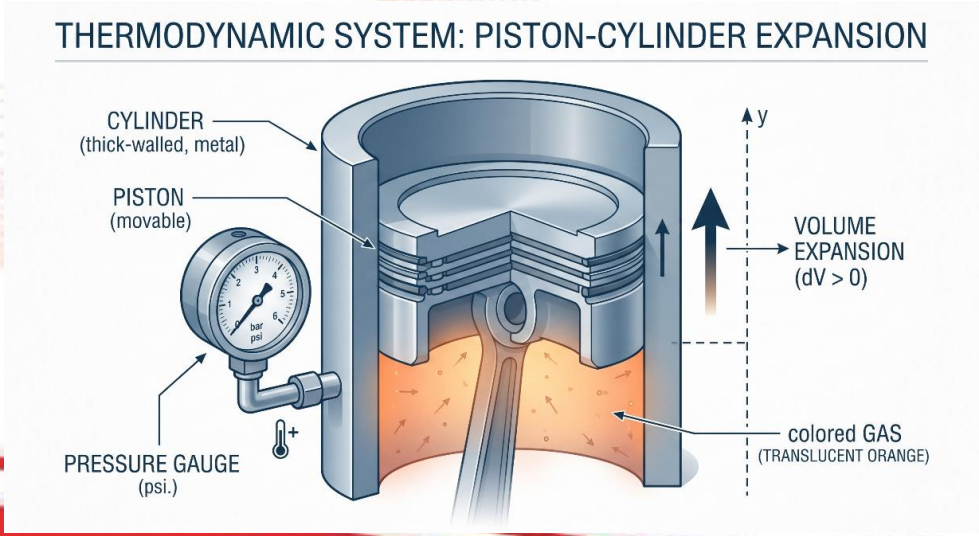

Figure 2.

Problem: A quantity of gas with an initial volume of 2m^3 and a pressure of 300kPa expands isothermally to a volume of 6m^3. Find the work done by the gas.

Solution:

1. Determine the constant C from the initial state:

$$C = P_1V_1 = (300\times10^3)\cdot2 = 600\times10^3 \text{ Joules}$$

2. Express pressure as a function of volume: $P(V) = \dfrac{600,000}{V}$.

3. Calculate the work by integrating from $V = 2$ to $V = 6$:

$$W = \int_2^6 \frac{600,000}{V}dV = 600,000\left[\ln|V|\right]_2^6 = 600,000(\ln 6 - \ln 2)$$

$$W = 600,000\ln\left(\frac{6}{2}\right) = 600,000\ln(3) \approx 659,160 \text{Joules} (or 659.16\text{kJ})$$

Here, the integral perfectly captures the inverse relationship between pressure and volume.

In fluid mechanics, the pressure exerted by a fluid at a depth h is $P = \rho gh$, where ρ is the fluid density and g is the gravitational acceleration. For a flat horizontal surface, the total force is simply pressure multiplied by area. However, for a vertical surface (like a dam), pressure varies continuously from the top to the bottom.

To find the total hydrostatic force F on a vertically submerged plate, we slice the plate into horizontal strips of infinitesimal thickness dy, width $w(y)$, at depth y. The force on each strip is $dF = (\rho gy)\cdot w(y)dy$. Integrating this from the top depth a to bottom depth b gives:

$$F = \int_a^b \rho gy\cdot w(y)dy$$

Example 3: Hydrostatic Force on a Rectangular Dam

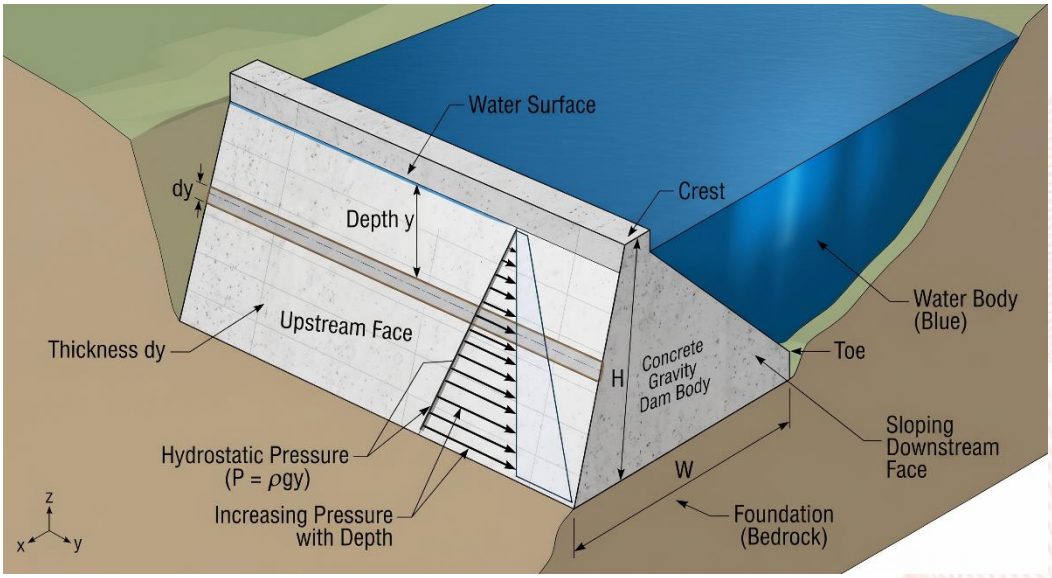

Figure 3.

Problem: A rectangular concrete dam is 10 meters wide and holds back water that is 5 meters deep. Calculate the total hydrostatic force exerted by the water on the dam. (Assume water density $\rho = 1000\text{kg/m}^3$ and $g = 9.8\text{m/s}^2$).

Solution:

1. We set up a coordinate system where $y = 0$ is the water surface, and the y-axis points downward.

2. The depth is y, varying from 0 to 5. The width of the dam is constant, so $w(y) = 10$.

3. The infinitesimal area of a horizontal strip is $dA = 10dy$. The pressure at depth y is $P(y) = 1000 \cdot 9.8 \cdot y = 9800y$.

4. The total force is the integral of pressure over the area:

$$F = \int_0^5 9800y \cdot (10)dy = 98000\int_0^5 ydy$$

$$F = 98000\left[\frac{y^2}{2}\right]_0^5 = 98000\left(\frac{25}{2} - 0\right) = 49000 \cdot 25 = 1,225,000\text{Newtons}$$

The definite integral allows civil engineers to design dams that can safely withstand these immense distributed forces.

The definite integral is a profoundly powerful instrument in physics and engineering, extending far beyond the calculation of simple geometric areas. By enabling the accumulation of continuously varying quantities—whether it is an increasing spring force, decreasing gas pressure, or depth-dependent fluid pressure—integral calculus provides exact analytical solutions necessary for safe and efficient engineering designs. Integrating these applications into higher mathematics education reinforces the critical link between abstract mathematical concepts and real-world physical phenomena.

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